18.06 FINAL - SOLUTIONS

PROBLEM 1

(1) Compute the determinant of $\begin{bmatrix} -1 & 2 & 5 & -3 \\ 1 & -3 & -3 & 2 \\ -2 & 7 & 8 & -5 \\ -4 & 7 & 2 & -1 \end{bmatrix}$ by row reduction/product of pivots.

Show all the steps. *(10 pts)*

Solution: Let's apply Gaussian elimination to row reduce the matrix:

$$\begin{bmatrix} -1 & 2 & 5 & -3 \\ 1 & -3 & -3 & 2 \\ -2 & 7 & 8 & -5 \\ -4 & 7 & 2 & -1 \end{bmatrix} \xrightarrow{r_2+r_1} \begin{bmatrix} -1 & 2 & 5 & -3 \\ 0 & -1 & 2 & -1 \\ -2 & 7 & 8 & -5 \\ -4 & 7 & 2 & -1 \end{bmatrix} \xrightarrow{r_3-2r_1} \begin{bmatrix} -1 & 2 & 5 & -3 \\ 0 & -1 & 2 & -1 \\ 0 & 3 & -2 & 1 \\ -4 & 7 & 2 & -1 \end{bmatrix} \xrightarrow{r_4-4r_1}$$

$$\begin{bmatrix} -1 & 2 & 5 & -3 \\ 0 & -1 & 2 & -1 \\ 0 & 3 & -2 & 1 \\ 0 & -1 & -18 & 11 \end{bmatrix} \xrightarrow{r_3+3r_2} \begin{bmatrix} -1 & 2 & 5 & -3 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 0 & -1 & -18 & 11 \end{bmatrix} \xrightarrow{r_4-r_2} \begin{bmatrix} -1 & 2 & 5 & -3 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & -20 & 12 \end{bmatrix} \xrightarrow{r_4+5r_3}$$

$$\begin{bmatrix} \boxed{-1} & 2 & 5 & -3 \\ 0 & \boxed{-1} & 2 & -1 \\ 0 & 0 & \boxed{4} & -2 \\ 0 & 0 & 0 & \boxed{2} \end{bmatrix}$$

The determinant is the product of the pivots, so $(-1) \cdot (-1) \cdot 4 \cdot 2 = 8$.

(2) Compute the determinant of $\begin{bmatrix} 0 & -2 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 \\ 4 & 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 3 & 0 \\ 0 & 0 & 3 & 0 & -1 \end{bmatrix}$ by cofactor expansion.

Show all the steps. *(10 pts)*

Solution: Let's do cofactor expansion along the third row (any other row or column would work equally well):

$$(-1)^{3+1}4 \cdot \det \begin{bmatrix} -2 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 7 & 3 & 0 \\ 0 & 3 & 0 & -1 \end{bmatrix} + (-1)^{3+4}5 \cdot \det \begin{bmatrix} 0 & -2 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}$$

(it's acceptable to note that the second determinant has a full column of zeroes, so it is 0, but I will compute it nonetheless for completeness). For each of the two determinants above, let's do cofactor expansion along the first row:

$$(-1)^{3+1}4 \cdot \left((-1)^{1+1}(-2) \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 3 & 0 & -1 \end{bmatrix} + (-1)^{1+4}1 \cdot \det \begin{bmatrix} -1 & 1 & 0 \\ 0 & 7 & 3 \\ 0 & 3 & 0 \end{bmatrix} \right) +$$

$$+(-1)^{3+4}5 \cdot \left((-1)^{1+2}(-2) \cdot \det \begin{bmatrix} 0 & 1 & 0 \\ 0 & 7 & 0 \\ 0 & 3 & -1 \end{bmatrix} + (-1)^{1+4}1 \cdot \det \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & 3 \end{bmatrix} \right)$$

Finally, we may compute the four determinants above by cofactor expansion along rows 1, 3, 1, 3, respectively:

$$(-1)^{3+1}4 \cdot \left((-1)^{1+1}(-2) \cdot (-1)^{1+1}1 \cdot \det \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} + (-1)^{1+4}1 \cdot (-1)^{3+2}3 \cdot \det \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \right) +$$

$$+(-1)^{3+4}5 \cdot \left((-1)^{1+2}(-2) \cdot (-1)^{1+2}1 \cdot \det \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + (-1)^{1+4}1 \cdot (-1)^{3+3}3 \cdot \det \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \right) =$$

$$(-1)^{3+1}4 \cdot (6 + (-9)) + (-1)^{3+4}5 \cdot (0 + 0) = -12$$

For the remainder of this problem, consider the following 10×10 matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(3) This matrix has two eigenvalues: just by inspecting the matrix, eyeball/guess what they are. Characterize the subspaces of eigenvectors corresponding to each eigenvalue. *(10 pts)*

Characterizing each of the two subspaces above can be done by either giving a basis of it, or by describing it implicitly as "the subspace of vectors satisfying the equation blah".

Solution: Any vector $\begin{bmatrix} x_1 \\ \vdots \\ x_{10} \end{bmatrix}$ that satisfies $x_1 + \cdots + x_{10} = 0$ is in the nullspace of A, which is just the subspace of eigenvectors corresponding to the eigenvalue 0. Another eigenvector is $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, since:

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and so the other eigenvalue is 10.

(4) What are the geometric multiplicities of the eigenvalues of the matrix A from the previous part? What are the algebraic multiplicities? What is the characteristic polynomial?

Explain how you know. *(15 pts)*

Solution: The subspace $\{x_1 + \cdots + x_{10} = 0\}$ has dimension 9 (10 degrees of freedom minus 1 constraint), so the geometric multiplicity of the eigenvalue 0 is 9. The subspace spanned by the eigenvector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is one-dimensional, so the geometric multiplicity of the eigenvalue 10 is 1. Since $9 + 1 = 10 =$ the size of the matrix, we conclude that the geometric multiplicities of the eigenvalues are as large as they can be. Therefore, they are also equal to the algebraic multiplicities, and so the characteristic polynomial is:

$$p(\lambda) = (-\lambda)^9(10 - \lambda) = \lambda^{10} - 10\lambda^9$$

(5) Based on the previous parts, what is the determinant of the matrix:

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{bmatrix}$$

Explain how you know. *(5 pts)*

Solution: We have $B = A + I$ and so:

$$\det B = \det(A + I) = p(-1) = (-1)^{10} - 10(-1)^9 = 11$$

PROBLEM 2

Throughout this problem, the matrix A has the following Singular Value Decomposition:

$$A = \underbrace{\frac{1}{3}\begin{bmatrix} 2 & 2 & -1 \\ x & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & y \end{bmatrix}}_{V^T}$$

where the matrices U and V are orthogonal and x, y denote two mystery real numbers.

The matrices U and V include the prefactors $\frac{1}{3}$ and $\frac{1}{5}$, so the top-left entry of U is $\frac{2}{3}$ and the top-left entry of V is $\frac{4}{5}$; recall that orthogonal means that $U^T U = I_3$ and $V^T V = I_2$.

(1) What are the values of x, y based on the information provided? Explain how you know. *(5 pts)*

Solution: We must have $x = -1$ and $y = 4$ because the matrices U and V are orthogonal, i.e. have orthonormal columns. So, for example, the dot product of columns 1 and 2 of U is:

$$\frac{1}{9}\left(2 \cdot 2 + x \cdot 2 + 2 \cdot (-1)\right) = 0$$

which gives us $x = -1$.

(2) Fill in the blanks (no explanation needed):

- the rank of the matrix A is $\underline{2}$ *(5 pts)*
- the eigenvalues of $A^T A$ are $\underline{9,1}$, and those of AA^T are $\underline{9,1,0}$ *(5 pts)*
- a non-zero eigenvector of AA^T is $\underline{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}}$ (any one eigenvector will suffice) *(5 pts)*

(3) Write A as a sum of two rank 1 matrices. *(5 pts)*

It suffices to write these rank 1 matrices as <u>*a column times a scalar times a row*</u>*, namely* $\mathbf{u} \cdot \sigma \cdot \mathbf{v}^T$*; you don't need to explicitly multiply the column, scalar and row out.*

Solution: The SVD gives us:

$$A = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \cdot 3 \cdot \frac{1}{5}\begin{bmatrix} 4 & -3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot 1 \cdot \frac{1}{5}\begin{bmatrix} 3 & 4 \end{bmatrix}$$

(4) Without computing A out explicitly, calculate the vector:

$$A\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Explain how you know. *(5 pts)*

Solution: We have $\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 5\boldsymbol{v}_1$ where $\boldsymbol{v}_1$, namely the first column on V, is the first right singular vector of the matrix A. Therefore, the fact that:

$$A\boldsymbol{v}_1 = \sigma_1 \boldsymbol{u}_1 \qquad \text{implies} \qquad A\begin{bmatrix} 4 \\ -3 \end{bmatrix} = 5 \cdot 3 \cdot \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

(5) What is the maximum of the following quantity as $\mathbf{v}$ ranges over non-zero vectors in $\mathbb{R}^2$:

$$\frac{||A\mathbf{v}||}{||\mathbf{v}||}$$

and for what $\mathbf{v}$ is it achieved? *(10 pts)*

Here $||\mathbf{v}||$ *denotes the length of the vector* $\mathbf{v}$*. Note that there is a whole line of* $\mathbf{v}$*'s for which the maximum is achieved; you simply need to find one non-zero vector on this line.*

Solution: The maximum is the largest singular value of A, namely 3, and its achieved for the first left singular vector of A, namely:

$$\frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

(6) Compute the pseudo-inverse A^+ of A, and explain how you got it (your answer for A^+ should be a 2×3 matrix with explicit numbers as entries). *(5 pts)*

Solution: The pseudo-inverse is given by:

$$A = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} = \frac{1}{45}\begin{bmatrix} 26 & 14 & -1 \\ 18 & 27 & -18 \end{bmatrix}$$

(7) Use A^+ to compute a least squares solution to $A\mathbf{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ (i.e. you must find a vector $\mathbf{v} \in \mathbb{R}^2$ such that $A\mathbf{v}$ is as close as possible to $\begin{bmatrix}1\\1\\1\end{bmatrix}$; explain which formula you are using). *(5 pts)*

Solution: The least squares solution in question is:

$$A^+ \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{15}\begin{bmatrix}13\\9\end{bmatrix}$$

PROBLEM 3

In this problem, we will consider the matrix:

$$X = \begin{bmatrix}1 & -1 & 1 & -1\\ 1 & 1 & 1 & 1\\ 1 & 0 & \alpha & 0\end{bmatrix}$$

for some mystery real number α.

(1) What does α need to be so that the matrix has rank 2? How do you know? *(10 pts)*

Solution: the rank is equal to the dimension of the row/column space. Since the first and second columns of X are clearly linearly independent, having rank 2 would require the third column to be a linear combination of the first two. In other words:

$$\begin{bmatrix}1\\1\\\alpha\end{bmatrix} = x\begin{bmatrix}1\\1\\1\end{bmatrix} + y\begin{bmatrix}-1\\1\\0\end{bmatrix}$$

Equating the coefficients reads $x - y = x + y = 1$ (which can be easily solved to give $x = 1$ and $y = 0$) and $\alpha = x$. We conclude that $\alpha = 1$.

Note that we would accept "eyeballing" the answer $\alpha = 1$, as long as its clear that the student meant that the third column is supposed to be a linear combination of the others.

(2) For the specific value of α from part (1), write down bases for the column space and the row space of X. (*Don't forget that there are as many vectors in a basis as the rank*) *(10 pts)*

Solution: As we've seen in the previous solution, a basis for the column space is given by:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Similarly, a basis for the row space is given by the first two rows:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

because they are linearly independent (also, the third row is a linear combination of the first two rows, specifically $\frac{1}{2}$ times the first row plus $\frac{1}{2}$ times the second row).

(3) Use the Gram-Schmidt process, and your result from part (2), to write down bases for:

- the left nullspace of X
- the nullspace of X

(α is still the specific number from part (1), which ensures that X has rank 2). *(20 pts)*

Solution: The left nullspace is a linear subspace of $\mathbb{R}^3$, explicitly defined as the orthogonal complement of the column space. Thus, to get a basis of the left nullspace, we need to complete an orthogonal basis of the column space to an orthogonal basis of $\mathbb{R}^3$; the vectors we add in this "completion" will be the required basis of the left nullspace.

So let's construct an orthogonal basis of $\mathbb{R}^3$, starting from the chosen basis of the column space that we found in the previous part plus a random third vector:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 13 \\ 9 \\ 20 \end{bmatrix}$$

(you must show why the third vector is not a linear combination of the first two, e.g. by computing the determinant of a 3×3 matrix, or an ad hoc attempt to write it as a linear combination of the first two vectors, with an explanation of why it fails). First we rescale $\boldsymbol{v}_1$ to make it have length 1:

$$\boldsymbol{q}_1 = \frac{\boldsymbol{v}_1}{||\boldsymbol{v}_1||} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then we subtract a multiple of $\boldsymbol{q}_1$ from $\boldsymbol{v}_2$ so as to make them orthgonal (this step is unnecessary in this case because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are already orthogonal, but let's do it anyway):

$$\boldsymbol{w}_2 = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 = \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Then we rescale $\boldsymbol{w}_2$ to make it have length 1:

$$\boldsymbol{q}_2 = \frac{\boldsymbol{w}_2}{||\boldsymbol{w}_2||} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Next, we subtract a linear combination of $\boldsymbol{q}_1, \boldsymbol{q}_2$ from $\boldsymbol{v}_3$ to make them orthogonal:

$$\boldsymbol{w}_3 = \boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{q}_1)\boldsymbol{q}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{q}_2)\boldsymbol{q}_2 = \boldsymbol{v}_3 - 14\sqrt{3}\boldsymbol{q}_1 + 2\sqrt{2}\boldsymbol{q}_2 = \begin{bmatrix} 13 - 14 - 2 \\ 9 - 14 + 2 \\ 20 - 14 + 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 6 \end{bmatrix}$$

Finally, we rescale $\boldsymbol{w}_3$ to make it have length 1:

$$\boldsymbol{q}_3 = \frac{\boldsymbol{w}_3}{||\boldsymbol{w}_3||} = \frac{1}{\sqrt{54}} \begin{bmatrix} -3 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Since $\boldsymbol{q}_1, \boldsymbol{q}_2$ form an (orthogonal) basis of the column space, their orthogonal complement (i.e. the left nullspace) is spanned by $\boldsymbol{q}_3$.

Similarly, the nullspace is a linear subspace of $\mathbb{R}^4$, explicitly defined as the orthogonal complement of the row space. Let's do the same procedure as in the paragraphs above: take any basis of the row space (e.g. the first two rows, as we saw in part (2)), complete them to an arbitrary basis of $\mathbb{R}^4$, and apply Gram-Schmidt. The resulting vectors that are not among the basis of the row space will be among the basis of the nullspace. So let's start from:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

(explain why the four vectors above form a basis). I won't go through the entire process, since you're likely to have started off from different $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ which will affect the final answer, but we expect you to go through the same process as in the previous paragraphs ($\boldsymbol{v} \rightsquigarrow \boldsymbol{w} \rightsquigarrow \boldsymbol{q}$). With the choices we made above, we get:

$$\boldsymbol{q}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_4 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Thus a basis of the nullspace is given by $\boldsymbol{q}_3$ and $\boldsymbol{q}_4$.

(4) Compute the general solution of the system of equations:

$$X\mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

where α is still the specific number from part (1), which ensures that X has rank 2. *(10 pts)*

Solution: The general solution is equal to:

$$\boldsymbol{v}_{\text{general}} = \boldsymbol{v}_{\text{particular}} + \boldsymbol{w}$$

where $\boldsymbol{w}$ is a general element in the nullspace of X. Since we have already computed a basis for the nullspace of X (namely $\boldsymbol{q}_3$ and $\boldsymbol{q}_4$ in part (3)), we have:

$$\boldsymbol{w} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

for arbitrary numbers α and β. As for a particular solution, we need to find numbers a, b, c, d such that:

$$X \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \qquad \Leftrightarrow \qquad a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Since a, b, c, d can be arbitrary, we might as well pick $c = d = 0$. Then the equation above compels us to have $a = 1$, and then by back substitution $b = 1$. We conclude that the general solution to the system of equations is:

$$\boldsymbol{v}_{\text{general}} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

for arbitrary numbers α and β.

PROBLEM 4

(1) Write the complex number i in polar form:

$$i = re^{i\theta}$$

where r is a positive real number, and θ is an angle. **Explain your answer**. *(5 pts)*

Solution: The absolute value and argument of the complex number $i = 0 + 1 \cdot i$ are given by the equations:

$$r = \sqrt{0^2 + 1^2} = 1 \qquad \text{and} \qquad \cos\theta = 0, \ \sin\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

We conclude that $i = e^{\frac{i\pi}{2}}$.

(2) Write the integer powers i^k (for <u>all</u> $k \in \mathbb{Z}$) both in:

- Cartesian form $a + bi$;
- polar form $re^{i\theta}$.

Explain your answer. *(10 pts)*

Solution: Since $i^4 = 1$, we have:

$$i^k = \begin{cases} 1 & \text{if } k = 4n \\ i & \text{if } k = 4n+1 \\ -1 & \text{if } k = 4n+2 \\ -i & \text{if } k = 4n+3 \end{cases}$$

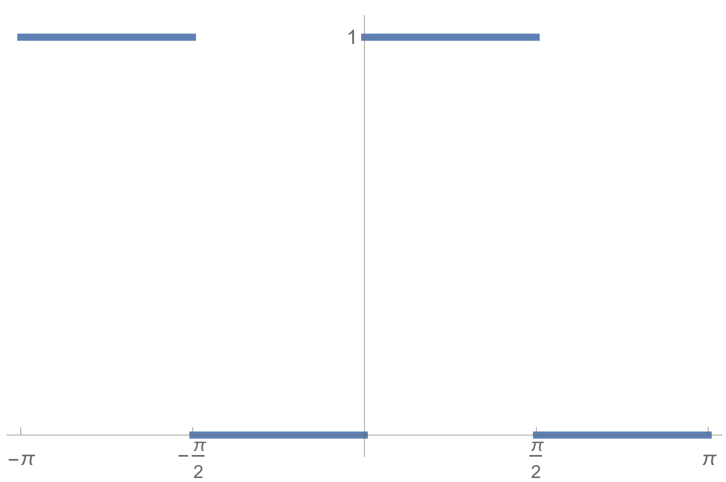
in Cartesian form. Meanwhile, in polar form, we have:

$$i^k = e^{\frac{ik\pi}{2}}$$

Consider the following "square" wave function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \left[-\pi, -\frac{\pi}{2}\right) \text{ or } x \in \left[0, \frac{\pi}{2}\right) \\ 0 & \text{if } x \in \left[-\frac{\pi}{2}, 0\right) \text{ or } x \in \left[\frac{\pi}{2}, \pi\right) \end{cases}$$

and extended to all $x \in \mathbb{R}$ by periodicity: $f(x + 2\pi) = f(x)$.



(3) Compute its <u>complex</u> Fourier series:

$$(*) \qquad f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$$

by which I mean, compute the coefficients c_k. **Simplify as much as possible**. *(10 pts)*

Solution: The Fourier coefficients are given by the formula:

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\left(\int_{-\pi}^{-\frac{\pi}{2}} e^{-ikx}dx + \int_0^{\frac{\pi}{2}} e^{-ikx}dx\right)$$

We clearly have $c_0 = \frac{1}{2}$, while for $k \neq 0$ we have:

$$c_k = \frac{1}{2\pi}\left(\frac{e^{-ikx}}{-ik}\Big|_{-\pi}^{-\frac{\pi}{2}} + \frac{e^{-ikx}}{-ik}\Big|_0^{\frac{\pi}{2}}\right) = \frac{e^{\frac{ik\pi}{2}} - e^{ik\pi} + e^{-\frac{ik\pi}{2}} - 1}{-2\pi ik}$$

We have $e^{i\pi} = -1$ and $e^{\frac{i\pi}{2}} = i$, so the formula above reads:

$$c_k = \frac{1 + (-1)^k - i^k - i^{-k}}{2\pi ik} = \frac{1 + (-1)^k - 2\mathrm{Re}(i^k)}{2\pi ik}$$

Since i^k is equal to 1, i, -1 or $-i$, depending on whether k is equal to $4n+0$, 1, 2 or 3, respectively, it is easy to see that:

$$c_k = \begin{cases} \frac{2}{\pi ik} & \text{if } k = 4n+2 \\ 0 & \text{otherwise} \end{cases}$$

We conclude that:

$$f(x) = \frac{1}{2} + 2\sum_{k=4n+2} \frac{e^{ikx}}{\pi ik}$$

(4) Compute the <u>real</u> Fourier series of f (using sines and cosines) by applying the formula:

$$e^{ix} = \cos x + i\sin x$$

to the right hand side of formula $(*)$ in part (3). **Simplify as much as possible**. *(10 pts)*

Solution: Replacing x by kx in the formula above gives us $e^{ikx} = \cos kx + i\sin kx$. So the Fourier series is given by:

$$f(x) = \frac{1}{2} + 2\sum_{k=4n+2} \frac{\cos kx + i\sin kx}{\pi ik}$$

To bring the formula above into real form, we need to combine the term corresponding to $k > 0$ with the term corresponding to $-k < 0$:

$$f(x) = \frac{1}{2} + 2\sum_{k=4n+2, n\geq 0}\left(\frac{\cos kx + i\sin kx}{\pi ik} + \frac{\cos kx - i\sin kx}{-\pi ik}\right) = \frac{1}{2} + 4\sum_{k=4n+2, n\geq 0} \frac{\sin kx}{\pi k}$$

(5) Using parts (1)-(4), compute the value of the Fourier series from $(*)$ at $x = \frac{\pi}{2}$, i.e.:

$$\sum_{k\in\mathbb{Z}} c_k e^{ik\frac{\pi}{2}}$$

where c_k are the coefficients you computed in part (1). *(5 pts)*

Solution: We have:

$$\frac{1}{2} + 2 \sum_{k=4n+2} \frac{e^{\frac{ik\pi}{2}}}{\pi ik} = \frac{1}{2} + 2 \sum_{k=4n+2} \frac{i^k}{\pi ik} = \frac{1}{2} - 2 \sum_{k=4n+2} \frac{1}{\pi ik} = \frac{1}{2}$$

The reason for the last equality is that the expression $\frac{1}{\pi ik}$ for a given $k > 0$ cancels out the expression $\frac{1}{-\pi ik}$ for $-k < 0$.

(6) How does the value of the Fourier series at $x = \frac{\pi}{2}$ compare to the value of $f\left(\frac{\pi}{2}\right)$? How do you explain this? *(10 pts)*

Solution: The value of the Fourier series, namely $\frac{1}{2}$, is precisely midway between the left and right limits of the function f near $x = \frac{\pi}{2}$. This makes sense, because if we simply changed the value of f at $x = \frac{\pi}{2}$ (without changing any of the other values) from 0 to 1, then this would not change the integrals that give rise to the Fourier coefficients.

PROBLEM 5

Throughout this problem, we will work with the matrix:

$$Z = \begin{bmatrix} 2 & -1 & -1 \\ 6 & -2 & -4 \\ 1 & -1 & 0 \end{bmatrix}$$

(1) Compute the characteristic polynomial of Z, and work out the eigenvalues. *(15 pts)*

Points will be taken off if you use the diagonals' method (a.k.a. Sarrus' rule) to compute 3×3 determinants. Use row reduction, cofactor expansion, or the big formula instead, and explain your process in detail.

Solution: the characteristic polynomial is:

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & -1 & -1 \\ 6 & -2-\lambda & -4 \\ 1 & -1 & -\lambda \end{bmatrix} = (2-\lambda)\cdot(-1)^{1+1} \det \begin{bmatrix} -2-\lambda & -4 \\ -1 & -\lambda \end{bmatrix} +$$

$$+(-1)\cdot(-1)^{1+2} \det \begin{bmatrix} 6 & -4 \\ 1 & -\lambda \end{bmatrix} + (-1)\cdot(-1)^{1+3} \det \begin{bmatrix} 6 & -2-\lambda \\ 1 & -1 \end{bmatrix} =$$

$$= (2-\lambda)(\lambda^2 + 2\lambda - 4) + (4 - 6\lambda) - (\lambda - 4) = -\lambda^3 + \lambda = -\lambda(\lambda^2 - 1) = -\lambda(\lambda - 1)(\lambda + 1)$$

So the eigenvalues are 1, 0 and -1.

(2) Diagonalize Z, i.e. write it as:

$$\boxed{Z = VDV^{-1}}$$

for an invertible matrix V and a diagonal matrix D. *(15 pts)*

Important for the following parts*: the diagonal entries of D should be the eigenvalues of Z, in order from the largest to the smallest.*

Solution: Let's compute eigenvectors for the eigenvalues 1,0, -1. I will show the process for the former, and then just give the answers for the latter. We need:

$$\boldsymbol{v}_1 \in N(Z - I) = N\left(\begin{bmatrix} 1 & -1 & -1 \\ 6 & -3 & -4 \\ 1 & -1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & -1 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

So we need $\boldsymbol{v}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where:

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \Leftrightarrow \qquad \begin{cases} x - y - z = 0 \\ 3y + 2z = 0 \\ 0 = 0 \end{cases}$$

We can pick $z = 1$ and then solve for the other variables by back substitution. We get:

$$\boldsymbol{v}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Similarly, for the other eigenvectors we get:

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

So we need to take $V = \begin{bmatrix} \frac{1}{3} & 1 & 1 \\ -\frac{2}{3} & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(3) For a general 3×3 matrix:

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

find matrices M and N such that:

$$MY = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \qquad \text{and} \qquad YN = \begin{bmatrix} y_{11} & y_{13} \\ y_{21} & y_{23} \\ y_{31} & y_{33} \end{bmatrix}$$

No explanation needed here. Pay attention to the indices! *(10 pts)*

Solution: We need to take $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(4) Using the previous parts of the problem, find 2×2 submatrices A, B and C of the 3×3 matrices V, D and V^{-1} (respectively) such that:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = ABC$$

"Submatrix" means that, for instance, A is obtained from V by removing one row and one column (you do not need to write A explicitly, but make sure you say which row/column you need to remove from V to get it; **explain your reasoning**) *(10 pts)*

Hint: look at the 2×2 *matrix* $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ *in relation to* Z.

Solution: We observe that the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ is obtained by deleting the second row and second column of Z. Therefore, by part (3), we have:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = MZN$$

But now let's replace Z by its diagonalization from part (2), so we have:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = MV \cdot D \cdot V^{-1}N$$

Suppose $V = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}$ and $V^{-1} = \begin{bmatrix} \overline{v}_{11} & \overline{v}_{12} & \overline{v}_{13} \\ \overline{v}_{21} & \overline{v}_{22} & \overline{v}_{23} \\ \overline{v}_{31} & \overline{v}_{32} & \overline{v}_{33} \end{bmatrix}$ (we can actually get numbers for these v_{ij} and $\overline{v}_{ij}$, but they will not be necessary to make this argument). So we have:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \overline{v}_{11} & \overline{v}_{13} \\ \overline{v}_{21} & \overline{v}_{23} \\ \overline{v}_{31} & \overline{v}_{33} \end{bmatrix}$$

However, because the second row and column of the middle matrix consists only of zeroes, all v's and $\overline{v}$'s with an index of 2 are not involved in the formula above. We conclude that:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{13} \\ v_{31} & v_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \overline{v}_{11} & \overline{v}_{13} \\ \overline{v}_{31} & \overline{v}_{33} \end{bmatrix}$$

So A, B, C are obtained by V, D, V^{-1} by removing the second row and second column.

PROBLEM 6

This problem consists of two parts: probability and statistics. While the language describing these two situations is often the same, the linear algebra tools used are different. So consider *the two as different problems from a mathematical standpoint.*

PROBABILITY: Paul the Octopus is not only good at predicting soccer scores, but he has magic powers. In his tank there are two boxes, one with the Dutch flag and one with the Spanish flag. Every time Paul creeps into one of the boxes, the respective soccer team scores a goal. However, for some reason Paul seems to prefer visiting the box with the Spanish flag k times more often than the box with the Dutch flag, where k is a natural number.

(1) What is, as a function of k, the probability that any one of Paul's visits is to the Dutch box? Same question for the Spanish box. *(5 pts)*

Solution: let p_s and p_d be the probabilities that Paul visits the Spanish and Dutch box, respectively. By assumption:

$$p_s = k \cdot p_d$$

However, probabilities must sum up to 1, so we have $p_s + p_d = 1$. Solving this system yields:

$$p_s = \frac{k}{k+1} \qquad \text{and} \qquad p_d = \frac{1}{k+1}$$

(2) Now suppose that Paul makes two consecutive (independent) visits to the boxes, according to the rule above. Consider the random vector:

$$\mathbf{v} = \begin{bmatrix} s \\ d \end{bmatrix}$$

where s (respectively d) keeps track of how many goals the Spanish (respectively Dutch) team scored as a consequence of these two visits. What are the possible values for the vector $\mathbf{v}$ and what are their probabilities? *(5 pts)*

Solution: The random vector $\mathbf{v}$ can take the value:

$\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ with probability $p_s^2 = \dfrac{k^2}{(k+1)^2}$ (S scores both goals)

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with probability $2p_s p_d = \dfrac{2k}{(k+1)^2}$ (S scores goal one, D scores goal two, or vice-versa)

$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ with probability $p_d^2 = \dfrac{1}{(k+1)^2}$ (D scores both goals)

(3) Compute the mean (i.e. the expected value) of the random vector $\mathbf{v}$. *(5 pts)*

Solution: We have:

$$\boldsymbol{\mu} = \frac{k^2}{(k+1)^2}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{2k}{(k+1)^2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{(k+1)^2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{2}{k+1}\begin{bmatrix} k \\ 1 \end{bmatrix}$$

(4) Compute the covariance matrix of the random vector $\mathbf{v}$. *(10 pts)*

Solution: We have:

$$K = \frac{k^2}{(k+1)^2}\left(\begin{bmatrix}2\\0\end{bmatrix} - \boldsymbol{\mu}\right)\left(\begin{bmatrix}2 & 0\end{bmatrix} - \boldsymbol{\mu}^T\right) + \frac{2k}{(k+1)^2}\left(\begin{bmatrix}1\\1\end{bmatrix} - \boldsymbol{\mu}\right)\left(\begin{bmatrix}1 & 1\end{bmatrix} - \boldsymbol{\mu}^T\right) +$$

$$+\frac{1}{(k+1)^2}\left(\begin{bmatrix}0\\2\end{bmatrix} - \boldsymbol{\mu}\right)\left(\begin{bmatrix}0 & 2\end{bmatrix} - \boldsymbol{\mu}^T\right) = \frac{2k}{(k+1)^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

(5) Diagonalize the covariance matrix computed in the previous bullet. Because the eigenvalues and eigenvectors are simple, you are allowed to simply guess them.

(10 pts)

Solution: Let's diagonalize the matrix $\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$. First, its characteristic polynomial is:

$$p(\lambda) = \det\begin{bmatrix}1-\lambda & -1\\-1 & 1-\lambda\end{bmatrix} = (1-\lambda)^2 - 1$$

The roots of this polynomial are $\lambda_1 = 2$ and $\lambda_2 = 0$. It's easy to show that eigenvectors corresponding to these two eigenvalues are:

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$$

So we have:

$$\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = VDV^{-1} \qquad \text{where } V = \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$$

The matrix K has the same diagonalization, but with D multiplied by $\frac{2k}{(k+1)^2}$.

(6) One of the eigenvalues of the covariance matrix should be 0. If:

$$\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

is the corresponding eigenvector, then the random variable $v_1 \cdot s + v_2 \cdot d$ should have variance 0. How do you explain this intuitively (i.e. based on the original probability setup)? *(5 pts)*

Solution: The eigenvector of 0 is $\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$, so the discussion above suggests that the random variable $s + d$ should have variance 0. This makes sense intuitively, because $s + d$ is equal to the total number of goals scored, which is equal to 2 no matter which boxes Paul visits.

STATISTICS: Suppose you have m sets consisting of n samples each. You may think of these sets as vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, and collect them as the columns of an $n \times m$ matrix:

$$\boldsymbol{A} = \left[\ \boldsymbol{a}_1 \,\middle|\, \ldots \,\middle|\, \boldsymbol{a}_m\ \right] = \begin{bmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \end{bmatrix}$$

The goal of principal component analysis (PCA) is to diagonalize the covariance matrix:

$$\boxed{K = \frac{\boldsymbol{A}^T P \boldsymbol{A}}{n-1} = QDQ^T}$$

where $P = \frac{1}{n} \begin{bmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{bmatrix} = I_n - \frac{1}{n} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$.

(7) Show that $P^2 = P$ and $P^T = P$. *(5 pts)*

(you may give either a geometric argument, or one via algebraic manipulations with matrices)

Solution: Let $R = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}$. Then it's easy to see that $R^2 = nR$, hence:

$$P^2 = \left(I_n - \frac{R}{n}\right)^2 = I_n - \frac{2R}{n} + \frac{R^2}{n^2} = I_n - \frac{R}{n} = P$$

Alternatively, $P^2 = P$ is a general property of projection matrices (and indeed our P is the projection onto the orthogonal complement of the vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$). The statement about P^T is obvious by direct computation; also, projection matrices are always symmetric.

(8) Suppose you had a machine which takes in any matrix, and produces the matrices U, Σ and V that give its singular value decomposition $U\Sigma V^T$. How can you use this machine to obtain the orthogonal matrix Q in the boxed formula on the previous page? *(5 pts)*

Solution: We have $K = \frac{\boldsymbol{A}^T P \boldsymbol{A}}{n-1} = \frac{\boldsymbol{A}^T P^2 \boldsymbol{A}}{n-1} = \frac{\boldsymbol{A}^T P^T P \boldsymbol{A}}{n-1} = \frac{(P\boldsymbol{A})^T (P\boldsymbol{A})}{n-1}$. Now apply our SVD machine to the matrix:

$$P\boldsymbol{A} = U\Sigma V^T$$

and we get:

$$K = \frac{1}{n-1} \cdot V\Sigma^T \underbrace{U^T U}_{I} \Sigma V^T = \frac{1}{n-1} \cdot V\Sigma^T \Sigma V^T$$

Since the matrix $\Sigma^T\Sigma$ is diagonal and V is orthogonal, we conclude that $V = Q$.

PROBLEM 7

(1) Fill in the blanks. The orthogonal complement of the vector:

$$\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

consists of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which satisfy the equation $\underline{x - z = 0}$ *(5 pts)*

(2) Choose basis vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ of the orthogonal complement of $\boldsymbol{a}$, and explain why $\boldsymbol{b}$, $\boldsymbol{c}$ form a basis. *Hint: your future will be easier if you pick* $\boldsymbol{b}$ *and* $\boldsymbol{c}$ *to be orthogonal.* *(10 pts)*

Solution: A basis of the plane $\{x - z = 0\}$ consists of any two vectors in this plane which are not proportional to each other. A possible choice is:

$$\boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(we chose these vectors such that $\boldsymbol{b} \perp \boldsymbol{c}$; the way you can do this is to choose $\boldsymbol{b}$ arbitrarily and then the three coefficients of $\boldsymbol{c}$ are constrained by two linear conditions: perpendicularity with $\boldsymbol{a}$ and with $\boldsymbol{b}$).

(3) Compute the projection matrix P onto the orthogonal complement of $\boldsymbol{a}$. *(10 pts)*

Solution: Consider the matrix:

$$A = \left[\ \boldsymbol{b} \mid \boldsymbol{c}\ \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the projection matrix is given by:

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

(the fact that we chose $\boldsymbol{b}$ and $\boldsymbol{c}$ to be perpendicular caused the 2×2 matrix in the formula above to be diagonal, which meant that it had a simple inverse).

(4) Consider the energy of an arbitrary vector with respect to the matrix P:

$$\begin{bmatrix} x & y & z \end{bmatrix} P \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Compute a formula for the energy and show that it is non-negative for any real numbers x, y, z. *(10 pts)*

Solution: We have:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x^2 + 2xz + z^2}{2} + y^2 = \frac{(x+z)^2}{2} + y^2$$

It is clearly non-negative for all x, y, z, as it is a sum of two squares.

(5) Find x, y, z (not all zero) for which the energy as above is 0. *(5 pts)*

Solution: Since the energy is a sum of two squares, it is 0 only when the individual squares are 0, so we need $x + z = y = 0$. Therefore, a choice of vector is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

(any multiple of this vector would have also worked).

(6) Fill in the blanks. The matrix P is positive <u>semi-definite</u> *(5 pts)*

(7) Fill in the blanks. The vector $\boldsymbol{a}$ and an arbitrary non-zero vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which the energy is 0 are <u>proportional</u>. *(5 pts)*

Although we accept synonyms of the word "proportional", we will deduct 2 points for the answer "equal", since the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *is only defined up to scalar multiple.*